Types for Hereditary Permutators

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Introduction

TLCA open problem 20:
- Typed Lambda Calculi and Applications
- Find a type system that characterizes hereditary permutators

Hereditary permutator
- a λ-term representing a bijection
- (infinite) nests of permutators

Results:
(1) No single type for hereditary permutators
- the set of hereditary permutators is not recursively enumerable
(2) Some countably infinite set of types for hereditary permutators

Ideas:
- coding of halting problem by an infinite Böhm tree
- intersection types for describing infinite computation

λ-Calculus

λ-terms $M, N, \ldots ::= x | \lambda x.M | MM$

β-reduction $(\lambda x.M)N \rightarrow_\beta M[x := N]$

β-equality $M =_\beta N$

M *head normal*
- if M is $\lambda x_1 \ldots x_n.yN_1 \ldots N_m$

M *head normalizing*
- if $M =_\beta N$ head normal

FV(M) the set of free variables in M
Λ the set of λ-terms

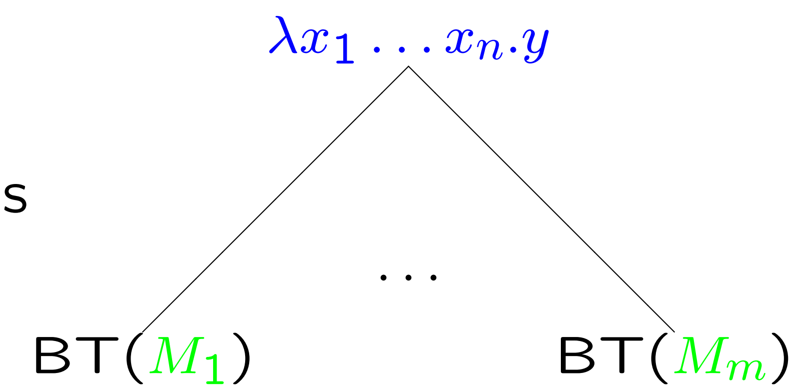
Böhm Tree

A (possibly infinite) tree with labels $\lambda x_1 \ldots x_n.y$ or $\bot$

Böhm tree BT(M) of a λ-term M is defined by

(1) BT(M) $= \bot$ if M not head normalizing

(2) BT(M) is



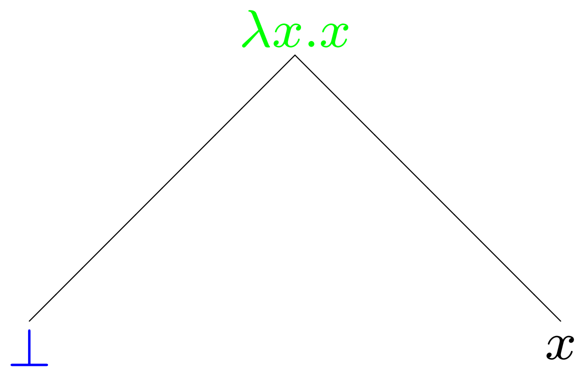
if $M =_\beta \lambda x_1 \ldots x_n.y M_1 \ldots M_m$

- represents infinite computation
- head variables   partial results
- $\bot$ useless computation

Examples of Böhm Trees

Let $\Delta = \lambda x.xx$

Eg 1. $\mathrm{BT}(\lambda x.x(\Delta\Delta)x) =$



Let $Y_0 = \lambda xy.y(xxy)$ and $Y = Y_0 Y_0$

Eg 2. $Yx =_\beta x(Yx) =_\beta x(x(Yx)) =_\beta \ldots$

$\mathrm{BT}(Yx) =$

Hereditary Permutators

A permutation Eg. (1 2 3 4 5) $\mapsto$ (1 3 2 5 4)

A permutator Eg. $f(x_1, x_2, x_3) \mapsto g(x_1, x_2, x_3) = f(x_2, x_3, x_1)$

This permutator is represented by $\lambda z x_1 x_2 x_3 . z x_2 x_3 x_1$
- $g = (\lambda z x_1 x_2 x_3 . z x_2 x_3 x_1) f$

Nests of permutators Eg. $f(x_1, x_2, x_3) \mapsto h(x_1, x_2, x_3) = f(x_2', x_3, x_1)$
where $x_2'(y_1, y_2) = x_2(y_2, y_1)$

This is represented by $\lambda z x_1 x_2 x_3 . z((\lambda z y_1 y_2 . z y_2 y_1) x_2) x_3 x_1$
- $h = (\lambda z x_1 x_2 x_3 . z((\lambda z y_1 y_2 . z y_2 y_1) x_2) x_3 x_1) f$

A hereditary permutator (infinite) nests of permutators

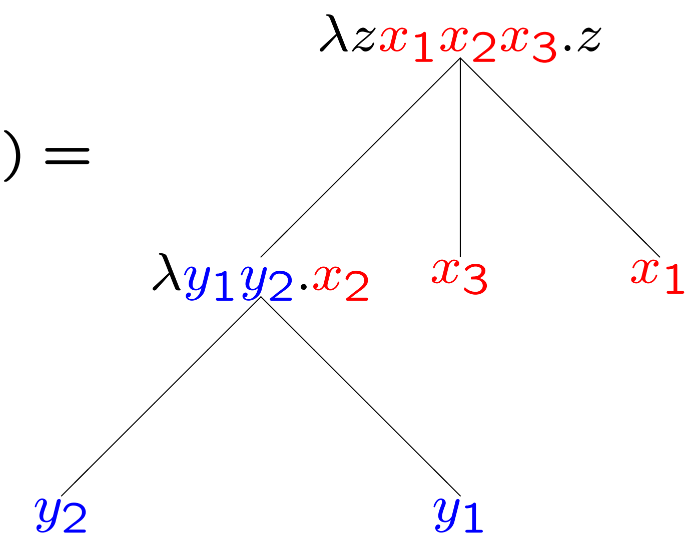
Definition of Hereditary Permutators

We call y the head variable of the node $\lambda x_1 \dots x_n . y$

λ-term M is hereditary permutator if BT(M) satisfies
(H1) Its root has the shape $\lambda z x_1 \dots x_n . z$, it has n child nodes, and each x_i is the head variable of some child node
(H2) A node except the root has the shape $\lambda x_1 \dots x_n . y$, it has n child nodes, and each x_i is the head variable of some child node

Eg.

BT($\lambda z x_1 x_2 x_3 . z((\lambda z y_1 y_2 . z y_2 y_1) x_2) x_3 x_1$) =



Related Work

M invertible

- if there is N such that $M(Nx) = x$ and $N(Mx) = x$
- a bijection

[Dezani 76]
Finite hereditary permutators are the same as invertible terms in $\lambda\beta\eta$

[Bergstra and Klop 80]
Hereditary permutators are the same as invertible terms in D_∞

A λ-term M is a hereditary permutator iff
M is a bijection in D_∞

Non-Recursive Enumerability

HP the set of hereditary permutators

Theorem. HP is not recursively enumerable

The next theorem immediately follows from this theorem

Theorem. There does not exist any type system T with any type A such that its language and the set of its inference rules are recursively enumerable, and HP is the same as $\{M \in \Lambda | \Gamma \vdash M : A \text{ is provable in } T \text{ for some } \Gamma\}$

Positive Primitive Recursive Functions

$\{e\}^{pr}(x)$ e-th unary primitive recursive function

PPR $= \{e \mid \forall x(\{e\}^{pr}(x) > 0)\}$
- the set of indices of positive primitive recursive functions

Theorem. PPR is not recursively enumerable

Proof. Any partial recursive function f is represented by
$f(x) = h(\mu y.(g(x, y) = 0))$ where g, h are primitive recursive

The index of $g(x, \_)$ is in PPR iff $f(x)$ is undefined

Hence PPR is not recursively enumerable $\square$

Primitive Recursive Functions in λ-Calculus

$\overline{n}$ n-th Church numeral $\lambda fx.f^n x = f(f(\ldots(fx)\ldots))$
Successor S $= \lambda yfx.f(yfx)$

Function $u(x,y) = \{x\}^{pr}(y)$
- a universal function for unary primitive recursive functions

λ-term U represents u
- $U\overline{n}\overline{m} =_\beta \overline{k}$ iff $u(n,m) = k$

Infinite Linear Hereditary Permutator

Infinite linear hereditary permutator $P = Y(\lambda p z_0 z_1 . z_0 (p z_1))$

$$\mathrm{BT}(P) = \begin{array}{c} \lambda z_0 z_1 . z_0 \\ | \\ \lambda z_2 . z_1 \\ | \\ \lambda z_3 . z_2 \\ | \\ \vdots \end{array}$$

Proof of Theorem

Let $T = Y(\lambda t x y z_0 z_1 . U x y (\lambda w . z_0 (t x (\mathsf{S} y) z_1))(\Delta\Delta))$

Then
$T \overline{e} \overline{n} z_n =_\beta \lambda z_1 . \Delta\Delta$ if $\{e\}^{pr}(n) = 0$
$T \overline{e} \overline{n} z_n =_\beta \lambda z_{n+1} . z_n (T \overline{e} \overline{(n+1)} z_{n+1})$ if $\{e\}^{pr}(n) > 0$

Hence
$e \in \mathrm{PPR}$ iff $\mathrm{BT}(\lambda z_0 . T \overline{e} \overline{0} z_0) = \mathrm{BT}(P)$

$$e \notin \mathrm{PPR} \text{ iff } \mathrm{BT}(\lambda z_0 . T \overline{e} \overline{0} z_0) = \begin{array}{c} \lambda z_0 z_1 . z_0 \\ | \\ \lambda z_2 . z_1 \\ | \\ \vdots \\ | \\ \lambda z_{m_0} . z_{m_0 - 1} \\ | \\ \bot \end{array}$$

where $\{e\}^{pr}(m_0) = 0$ and $\{e\}^{pr}(m) > 0$ for $m < m_0$

Therefore $e \in \mathrm{PPR}$ iff $\lambda z_0 . T \overline{e} \overline{0} z_0 \in \mathrm{HP}$
Hence HP is not recursively enumerable $\square$

A Best-Possible Solution

$M \in$ HP not represented by $\exists x P(M, x)$, but $\forall n \exists x P(M, n, x)$
where P quantifier-free

The next goal:
- Find p_n such that $M : p_n$ for all n iff $M \in$ HP
- (Actually HP is Π^0_2-complete)

A solution:
- $M : p_n$ iff BT(M) of depth $< n$ satisfies the conditions (H1) and (H2)
- Because $M \in$ HP iff BT(M) satisfies (H1) and (H2)

Type System $\mathcal{T}$

Type constants $p_n, q_m \quad (n \geq 0, m \geq 1)$, Ω

Types $A, B, \ldots ::= p_n | q_m | \Omega | A \rightarrow A | A \cap A$

TC($\vec{A}$) the set of type constants in $\vec{A}$

$\mathcal{S}_m$ the symmetric group of order m

Type partial equivalence $A \sim_n B$ for $n > 0$ is defined by

$$\Omega \sim_0 \Omega$$

$$\frac{A_i \sim_n B_i \qquad (1 \leq i \leq m)}{B_{\pi(1)} \rightarrow \ldots \rightarrow B_{\pi(m)} \rightarrow q_k \sim_{n+1} A_1 \rightarrow \ldots \rightarrow A_m \rightarrow q_k}$$

where $\pi \in \mathcal{S}_m$ and TC$(A_i, B_i) - \{\Omega\} \quad (1 \leq i \leq m), \{q_k\}$ are disjoint

Inference Rules

$$\frac{}{\Gamma, x : A \vdash x : A}\,(Ass) \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B}\,(\to I)$$

$$\frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B}\,(\to E)$$

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash M : B}{\Gamma \vdash M : A \cap B}\,(\cap I)$$

$$\frac{\Gamma \vdash M : A \cap B}{\Gamma \vdash M : A}\,(\cap E_1) \qquad \frac{\Gamma \vdash M : A \cap B}{\Gamma \vdash M : B}\,(\cap E_2)$$

$$\frac{}{\Gamma \vdash M : \Omega}\,(\Omega) \qquad \frac{\Gamma, z : A \vdash M : B \quad A \sim_n B}{\Gamma \vdash \lambda z.M : p_n}\,(p_n I)$$

Theorem. $\vdash M : p_n$ for all n iff $M \in$ HP

Permutator Scheme

$$\begin{aligned}
&\mathsf{PS}_0(z) = \Lambda \\
&\mathsf{PS}_{n+1}(z) = \{M \in \Lambda | \\
&\quad M =_\beta \lambda x_1 \ldots x_m.zM_{\pi(1)} \ldots M_{\pi(m)}, \\
&\quad \pi \in \mathcal{S}_m, M_i \in \mathsf{PS}_n(x_i) \quad (1 \le i \le m)\}
\end{aligned}$$

$M \in \mathsf{PS}_n(z)$

- $\mathsf{BT}(\lambda z.M)$ of depth $< n$ satisfies (H1) and (H2)

Lemma. $M \in \mathsf{PS}_n(z)$ for all n iff $\lambda z.M \in \mathsf{HP}$

Soundness Proof

right(A) the rightmost type constant in A

Proposition. If $\overrightarrow{x : B} \vdash M : A$ and right(A) $\neq \Omega$, M is head normalizing

This is proved by

$$[\![q_n]\!] = [\![p_{n+1}]\!] = \text{(head normalizing terms)}$$
$$[\![\Omega]\!] = \Lambda$$
$$[\![A \to B]\!] = [\![A]\!] \to [\![B]\!]$$
$$[\![A \cap B]\!] = [\![A]\!] \cap [\![B]\!]$$

Key Lemma. If $A \sim_n B$ and $\Gamma, z : A \vdash M : B$ are provable and core(Γ) $\cap$ (TC(A, B) $- \{\Omega\}$) $= \phi$, then M is in PS$_n$(z), where

$$\text{core}(c) = \{c\} \qquad (c = q_n, p_n, \Omega)$$
$$\text{core}(A \to B) = \text{core}(B)$$
$$\text{core}(A \cap B) = \text{core}(A) \cup \text{core}(B)$$

This is proved by induction on n

Lemma. $\vdash \lambda z.M : p_n$ implies $M \in$ PS$_n$(z)

Completeness Proof

Lemma. If $M \in \mathsf{PS}_n(z)$, there are A and B such that $z : A \vdash M : B$ and $A \sim_n B$

This is proved by induction on n

Example: Types for Linear Hereditary Permutators

Let $P = Y(\lambda fxy.x(fy))$

Then $\mathrm{BT}(P) =$

$$\begin{array}{c} \lambda x_0 x_1.x_0 \\ | \\ \lambda x_2.x_1 \\ | \\ \lambda x_3.x_2 \\ | \\ \vdots \end{array}$$

$P \in$ HP (infinite linear hereditary permutator)

Let $P_0 = \lambda z.z$ and $P_{n+1} = \lambda z x_1.z(P_n x_1)$

Then $\mathrm{BT}(P_n) =$

$$\begin{array}{c} \lambda z_0 z_1.z_0 \\ | \\ \lambda z_2.z_1 \\ | \\ \vdots \\ | \\ \lambda z_n.z_{n-1} \\ | \\ z_n \end{array}$$

$P_n \in$ HP (finite linear hereditary permutator)

Example (cont)

Let $A_0 = \Omega$
$A_{n+1} = A_n \rightarrow q_{n+1}$

Then $\vdash P : A_n \rightarrow A_n$ for all n
$\vdash P_m : A_n \rightarrow A_n$ for all n

If $\vdash M : A_n \rightarrow A_n$ for all n,
then $\mathrm{BT}(M) = \mathrm{BT}(P)$ or $M =_\beta P_m$ for some m

Conclusion

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