Program Analysis using Quantifier Elimination Heuristics

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(work in progress)

Outline

- Quantifier Elimination Approach for Generating (Loop) Invariants.

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- Octagonal Constraints.

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- A fast $O(n^2)$ to generate the strongest octagonal invariant
- Disjunctive Invariants – **max plus** constraints

Invariants: Integer Square Root

Example

```
x := 1, y := 1, z := 0;
while (x <= N) {
  x := x + y + 2;
  y := y + 2;
  z := z + 1
}
return z
```

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    $(I(x, y, z) \wedge x \leq N) \Longrightarrow I(x + y + 2, y + 2, z + 1).$

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    $(I(x, y, z) \wedge x \leq N) \Longrightarrow I(x + y + 2, y + 2, z + 1).$
- Using quantifier elimination, find constraints on parameters $A, B, C, D, E, F, G, H, J, K$ which ensure that the verification conditions are valid for all possible program variables.

- Constraints on parameters are:

$$C = -F, \;\; J = -2B - F + 2K, \;\; G = -4B - F, \;\; H = 3B + F - K.$$

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- Most general invariants describing all invariants are:

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from which $x = (z + 1)^2$ follows.

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    approximations of assignments and test conditions may be necessary.
- Find a formula expressed in terms of parameters eliminating all program variables (using quantifier elimination).

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  - If no parameter values can be found, then invariants of hypothesized forms may not exist. Invariants can be guaranteed not to exist if no approximations are made, while generating verification conditions.
- If all assignments making the formula true can be finitely described, invariants generated may be the strongest of the hypothesized form. Invariants generated are guaranteed to be the strongest if no approximations are made, while generating verification conditions.

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- Combination of Theories–Presburger Arithmetic with Theory of Equality over Uninterpreted Symbols (Shostak, 1979; Nelson, 1981), and with Boolean Algebra (Kuncak, 2007), etc.
- Reduction Approach to Decision Procedures for Theories over Abstract Data Structures, including Finite Lists, Finite Sets, Finite Arrays, Finite Multisets (Kapur and Zarba, 2005).

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- Extensively investigated in many areas including program analysis, program synthesis, termination of programs, as well as hybrid system analysis, particularly safety check and controller synthesis.
- Closely related to choosing an abstract domain in the abstract interpretation approach.

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  - Output is huge and difficult to decipher.
  - In practice, they often do not work (i.e., run out of memory or hang).
- Linear constraint solving on rationals and reals (polyhedral domain), while of polynomial complexity, has been found in practice to be inefficient and slow, especially when used repeatedly as in abstract interpretation approach [Miné]

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- Miné gave well-designed algorithms based on Difference Bound Matrices (DBMs) and graph representation for performing various operations needed for program analysis using the abstract interpretation approach.
- Miné's algorithms are of $O(n^3)$ (sometimes, $O(n^4)$), where n is the number of variables.

Octagonal Constraints and Quantifier Elimination

- Octagonal constraints have a fixed shape. Given n variables, the most general formula (after simplification) is of the following form
  $\bigwedge_{i,j}(\;\; I_{i,j} : \;\; a_{i,j} \leq x_i - x_j \leq b_{i,j}, \;\; c_{i,j} \leq x_i + x_j \leq d_{i,j}, \;\; e_i \leq x_i \leq f_i \;\; g_j \leq x_j \leq h_j)$

  for every pair of variables x_i, x_j, where $a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j}, e_i, f_i, g_j, h_j$ are parameters.

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- For a finite program path consisting of a sequence of assignment statements interspersed with tests, its behavior is approximated so that the post condition is also of the above form.
- A verification condition is expressed using atomic formulas that are all octagonal constraints.

$$\bigwedge_{i,j}((I_{i,j} \wedge \alpha(x_i, x_j)) \Rightarrow I'_{i,j}),$$

along with additional parameter-free constraints $\alpha(x_i, x_j)$, of the same form in which lower and upper bounds are constants.

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  Assignment statements are of the form $x_i := x_i + a$ or $x_i := -x_i + a$. And, tests are lower and upper bounds on variables and expressions of the form $\pm x_i \pm x_j$. Otherwise, tests and assignments must be approximated.

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- Quantifier elimination heuristics can be developed using which it is possible to generate constraints on lower and upper bounds by table look ups in $O(n^2)$ steps, where n is the number of program variables.

Geometric QE Heuristic

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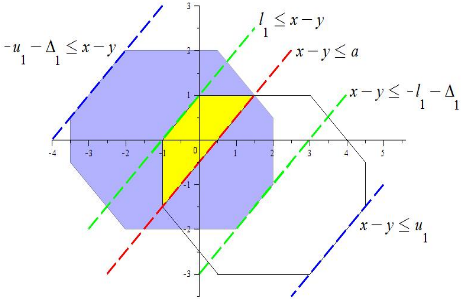
- Identify conditions under which the transformed octagon includes the portion of the original octagon satisfying tests along a program path.
- For each assignment case, a table is built showing the effect on the parameter values by determining the effect on every type of constraints.

Table 1: Assignments with signs of variables reversed

$$x_i := -x_i + A, \quad x_j := -x_j + B, \quad \Delta_1 = B - A, \ \ \Delta_2 = -A - B, \ \ \Delta_3 = -A, \ \ \Delta_4 = -B.$$



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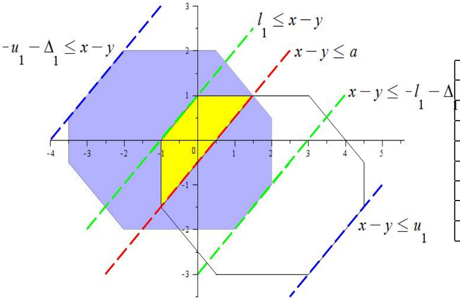

	present	absent
$x_i - x_j \le a$	$a \le -l_1 - \Delta_1$	$u_1 \le -l_1 - \Delta_1$
$x_i - x_j \ge b$	$-u_1 - \Delta_1 \le b$	$-u_1 - \Delta_1 \le l_1$
$x_i + x_j \le c$	$c \le -l_2 - \Delta_2$	$u_2 \le -l_2 - \Delta_2$
$x_i + x_j \ge d$	$-u_2 - \Delta_2 \le d$	$-u_2 - \Delta_2 \le l_2$
$x_i \le e$	$e \le -l_3 - \Delta_3$	$u_3 \le -l_3 - \Delta_3$
$x_i \ge f$	$-u_3 - \Delta_3 \le f$	$-u_3 - \Delta_3 \le l_3$
$x_j \le g$	$g \le -l_4 - \Delta_4$	$u_4 \le -l_4 - \Delta_4$
$x_j \ge h$	$-u_4 - \Delta_4 \le h$	$-u_4 - \Delta_4 \le l_4$

Table 2: No changing signs of variables

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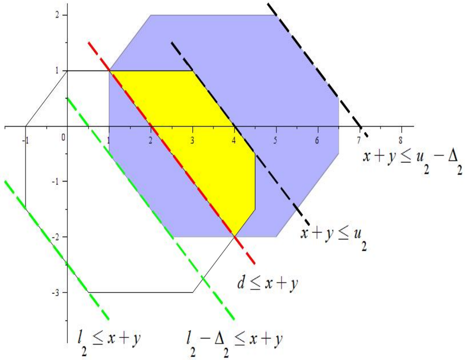

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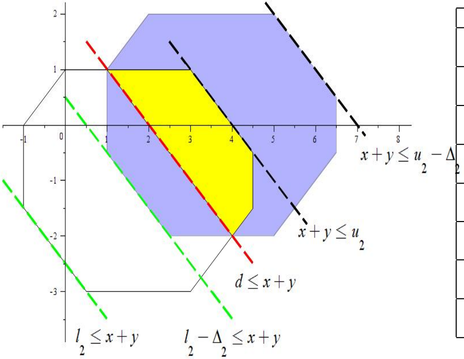
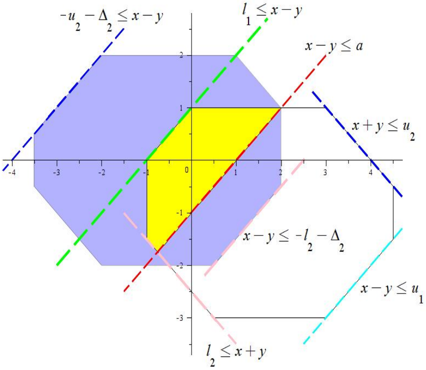


	present	absent
$x_i - x_j \leq a$ $\Delta_1 > 0$	$u_1 \geq a + \Delta_1$	$u_1 = +\infty$
$x_i - x_j \geq b$ $\Delta_1 < 0$	$l_1 \leq b + \Delta_1$	$l_1 = -\infty$
$x_i + x_j \leq c$ $\Delta_2 > 0$	$u_2 \geq c + \Delta_2$	$u_2 = +\infty$
$x_i + x_j \geq d$ $\Delta_2 < 0$	$l_2 \leq d + \Delta_2$	$l_2 = -\infty$
$x_i \leq e$ $\Delta_3 > 0$	$u_3 \geq e + \Delta_3$	$u_3 = +\infty$
$x_i \geq f$ $\Delta_3 < 0$	$l_3 \leq f + \Delta_3$	$l_3 = -\infty$
$x_j \leq g$ $\Delta_4 > 0$	$u_4 \geq g + \Delta_4$	$u_4 = +\infty$
$x_j \geq h$ $\Delta_4 < 0$	$l_4 \leq h + \Delta_4$	$l_4 = -\infty$

Table 3: Sign of exactly one variable is changed

$$x_i := -x_i + A, \quad x_j := x_j + B, \quad \Delta_1 = B - A, \quad \Delta_2 = -A - B, \quad \Delta_3 = -A, \quad \Delta_4 = B.$$



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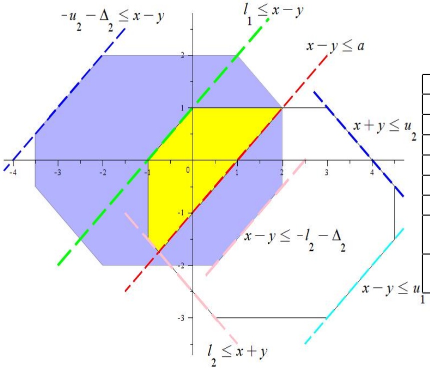

	present	absent
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$x_i + x_j \leq c$	$c \leq -l_1 - \Delta_1$	$u_2 \leq -l_1 - \Delta_1$
$x_i + x_j \geq d$	$-u_1 - \Delta_1 \leq d$	$-u_1 - \Delta_1 \leq l_2$
$x_i \leq e$	$e \leq -l_3 - \Delta_3$	$u_3 \leq -l_3 - \Delta_3$
$x_i \geq f$	$-u_3 - \Delta_3 \leq f$	$-u_3 - \Delta_3 \leq l_3$
$x_j \leq g$ $\Delta_4 > 0$	$u_4 \geq g + \Delta_4$	$u_4 = +\infty$
$x_j \geq h$ $\Delta_4 < 0$	$l_4 \leq h + \Delta_4$	$l_4 = -\infty$

A Simple Example

Example

```
 x := 4; y := 6;
while (x + y >= 0) do
 if (y >= 6) then { x := -x;  y := y - 1 }
    else { x := x - 1; y := -y }
endwhile
```

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VC0: $I(4, 6)$
VC1: $(I(x, y) \wedge (x + y) \geq 0 \wedge y \geq 6) \implies I(-x, y - 1)$.
VC2: $(I(x, y) \wedge (x + y) \geq 0 \wedge y < 6) \implies I(x - 1, -y)$.

Generating Constraints on Parameters

- **VC0:**
  $l_1 \leq -2 \leq u_1 \wedge l_2 \leq 10 \leq u_2 \wedge l_3 \leq 4 \geq u_3 \wedge l_4 \leq 6 \leq u_4.$

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- Making the l_i's as large as possible and u_i's as small as possible:

$$l_1 = -10, u_1 = 9, l_2 = -11, u_2 = 10, l_3 = -6, u_3 = 6, l_4 = -5, u_4 = 6$$

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- The corresponding invariant is:

  $$-10 \leq x-y \leq 9 \wedge -11 \leq x+y \leq 10 \wedge -6 \leq x \leq 6 \wedge -5 \leq y \leq 6.$$

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- Overall Complexity: $O(n^2)$.

Local QE Heuristics: Propagation of Tests?

- Local propagation (i.e., propagate bounds only for the pair of variables appearing in the constraints) on tests to bring them in a canonical form can sometimes improve the bounds. But that is not always clear. The complexity still remains $O(n^2)$.

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- There are examples for which both local and global propagation (closure) lead to worse results.

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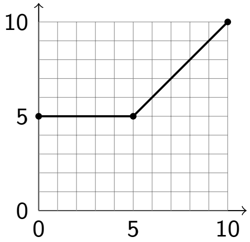
- It has been found in practice that QE based methods generate stronger inductive invariants than methods based on abstract interpretation for polyhedral domain.
- An open question: the strength of invariants generated by the proposed incomplete heuristics.

Disjunctive Invariants

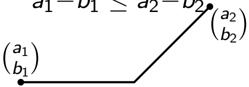
$x := 0; y := 5;$
while $(x < 10)$ do
  if $(x < 5)$ then
    $x := x + 1;$
  else
    x := x+1; y := y+1;



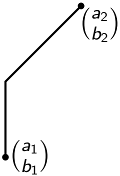
$I = (0 \leq x \leq 5 \wedge y = 5) \vee (5 \leq x \leq 10 \wedge x = y)$.
It consists of two lines and is clearly not convex.
It cannot be expressed as a conjunction of linear constraints, including octagonal constraints or even general polyhedra.

Shapes

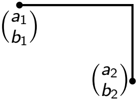
Shape 1:
$b_1 \leq b_2$
$a_1 - b_1 \leq a_2 - b_2$



Shape 2:
$b_1 \leq b_2$
$a_1 - b_1 \geq a_2 - b_2$



Shape 3:
$b_1 \geq b_2$



Shape 1

assignment statements	max-plus polyhedra to be cut off
$A = 0, B = 0$	–
$A > 0, B = 0, \quad A \leq \Delta_a - \Delta_b$	$\mathrm{co}(\{\binom{a_2-\Delta_b-A+1}{b_1}, \binom{a_2}{b_2}\})$
$A > 0, B = A, \quad A \leq \Delta_b$	$\mathrm{co}(\{\binom{a_1}{b_1}, \binom{a_2-\Delta_b-1}{b_1}\})$, if $\Delta_a \neq \Delta_b$ $\mathrm{co}(\{\binom{a_2-A+1}{b_2-A+1}, \binom{a_2}{b_2}\})$
$A > B, B > 0, \quad \Delta \leq \Delta_a - \Delta_b, B \leq \Delta_b$	$\mathrm{co}(\{\binom{a_1}{b_1}, \binom{a_2-\Delta_b-\Delta-1}{b_1}\})$, if $\Delta_b + \Delta \neq \Delta_a$ $\mathrm{co}(\{\binom{a_2}{b_2}, \binom{a_2-\Delta_b-\Delta+1}{b_1}\})$
$A < 0, B = 0, \quad \lvert A\rvert \leq \Delta_a - \Delta_b$	$\mathrm{co}(\{\binom{a_1}{b_1}, \binom{a_1-A-1}{b_1}\})$ $\mathrm{co}(\{\binom{a_2-\Delta_b+1}{b_1+1}, \binom{a_2}{b_2}\})$, if $\Delta_b \neq 0$
$A < 0, B = A, \quad \lvert A\rvert \leq \Delta_b$	$\mathrm{co}(\{\binom{a_1}{b_1}, \binom{a_2-\Delta_b-A+1}{b_1-A+1}\})$
$A < B, B < 0, \quad \Delta \leq \Delta_a - \Delta_b, \lvert B\rvert < \Delta_b$	$\mathrm{co}(\{\binom{a_1}{b_1}, \binom{a_2-\Delta_b-B-1}{b_1-B-1}\})$ $\mathrm{co}(\{\binom{a_2}{b_2}, \binom{a_2-\Delta_b-B+1}{b_1-B+1}\})$, if $B + \Delta_b \neq 0$
all other cases	$\mathrm{co}(\{\binom{a_1}{b_1}, \binom{a_2}{b_2}\})$

Shape 2

assignment statements		max-plus polyhedra to be cut off
$A = 0, B = 0$		–
$A = 0, B > 0,$	$B \leq \Delta_b - \Delta_a$	$\mathrm{co}(\{\binom{a_1}{b_2 - \Delta_a - B + 1}, \binom{a_2}{b_2}\})$
$A = 0, B < 0,$	$\lvert B \rvert \leq \Delta_b - \Delta_a$	$\mathrm{co}(\{\binom{a_1}{b_1}, \binom{a_1}{b_1 - B + 1}\})$ $\mathrm{co}(\{\binom{a_1 + 1}{b_2 - \Delta_a + 1}, \binom{a_2}{b_2}\})$, if $a_1 \neq a_2$
$A > 0, B = A,$	$A \leq \Delta_a$	$\mathrm{co}(\{\binom{a_1}{b_1}, \binom{a_1}{b_2 - \Delta_a - 1}\})$, if $\Delta_a \neq \Delta_b$ $\mathrm{co}(\{\binom{a_2}{b_2}, \binom{a_2 - A + 1}{b_2 - A + 1}\})$
$A > 0, B > A,$	$A \leq \Delta_a, \Delta \geq \Delta_b - \Delta_a$	$\mathrm{co}(\{\binom{a_1}{b_1}, \binom{a_1}{b_2 - \Delta_a + \Delta - 1}\})$ $\mathrm{co}(\{\binom{a_2}{b_2}, \binom{a_1}{b_2 - \Delta_a + \Delta + 1}\})$, if $\Delta_a - \Delta \neq \Delta_b$
$A < 0, B = A,$	$\lvert A \rvert \leq \Delta_a$	$\mathrm{co}(\{\binom{a_1}{b_1}, \binom{a_1 - A - 1}{b_2 - \Delta_a - A - 1}\})$
$A < 0, B < A,$	$\lvert A \rvert \leq \Delta_a, \Delta \leq \Delta_b - \Delta_a$	$\mathrm{co}(\{\binom{a_1}{b_1}, \binom{a_1 - A - 1}{b_2 - \Delta_a - A - 1}\})$ $\mathrm{co}(\{\binom{a_2}{b_2}, \binom{a_1 - A + 1}{b_2 - \Delta_a - A + 1}\})$, if $\Delta_b + A \neq 0$
all other cases		$\mathrm{co}(\{\binom{a_1}{b_1}, \binom{a_2}{b_2}\})$

Shape 3

assignment statements	max-plus polyhedra to be cut off
$A = 0, B = 0$	–
$A = 0, B > 0, \quad B \leq -\Delta_b$	$\mathrm{co}(\{\binom{a_1}{b_1}, \binom{a_2}{b_1 - B + 1}\})$
$A = 0, B < 0, \quad B \geq \Delta_b$	$\mathrm{co}(\{\binom{a_1}{b_1}, \binom{a_2 - 1}{b_1}\})$, if $a_1 \neq a_2$ $\mathrm{co}(\{\binom{a_2}{b_2}, \binom{a_2}{b_2 - B - 1}\})$
$A > 0, B = 0, \quad A \leq \Delta_a$	$\mathrm{co}(\{\binom{a_2}{b_2}, \binom{a_2 - A + 1}{b_1}\})$
$A > 0, B < 0, \quad A \leq \Delta_a, B \geq \Delta_b$	$\mathrm{co}(\{\binom{a_1}{b_1}, \binom{a_2 - A - 1}{b_1}\})$, if $A \neq \Delta_a$ $\mathrm{co}(\{\binom{a_2}{b_2}, \binom{a_2 - A + 1}{b_1}\})$
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$A < 0, B > 0, \quad \lvert A \rvert \leq \Delta_a, B \leq -\Delta_b$	$\mathrm{co}(\{\binom{a_1}{b_1}, \binom{a_2}{b_1 - B + 1}\})$ $\mathrm{co}(\{\binom{a_2}{b_2}, \binom{a_2}{b_1 - B - 1}\})$, if $\Delta_b + B \neq 0$.
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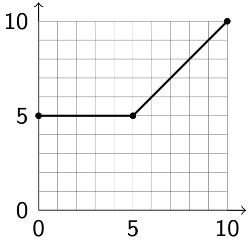
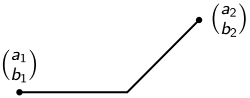
Example

$x := 0; y := 5;$
while $(x < 10)$ do
  if $(x < 5)$ then
    $x := x + 1;$
  else
    x := x+1; y := y+1;



Example

Assumed shape of M:



Loop invariant M must be satisfied when the loop is entered.
The initial value is situated on the horizontal branch of M giving the constraints
$a_1 \leq 0 \leq a_2 - (b_2 - b_1)$ and $b_1 = 0$.

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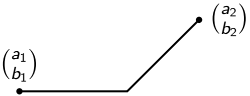

First branch: VC is

$$\binom{x}{y} \in \mathrm{co}(\{\binom{a_1}{b_1}, \binom{a_2}{b_2}\}) \wedge (x < 10 \wedge x < 5) \implies \binom{x+1}{y} \in \mathrm{co}(\{\binom{a_1}{b_1}, \binom{a_2}{b_2}\})$$

or equivalently

$$\binom{x}{y} \in \mathrm{co}(\{\binom{a_1}{b_1}, \binom{a_2}{b_2}\}) \wedge (x < 10 \wedge x < 5) \implies \binom{x}{y} \in \mathrm{co}(\{\binom{a_1-1}{b_1}, \binom{a_2-1}{b_2}\}) \,.$$

Resulting constraint: $a_2 - (b_2 - b_1) - 1 \geq 4$

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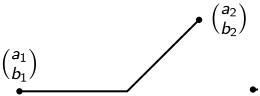
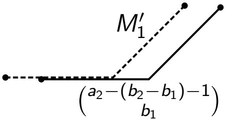


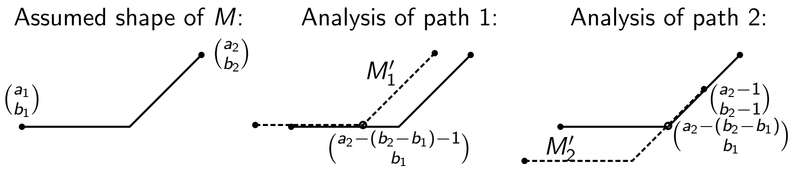
Analysis of path 1:



Second branch: let M_2' be the max-plus polyhedron generated by $\binom{a_1-1}{b_1-1}$ and $\binom{a_2-1}{b_2-1}$.

For M to be an invariant, every point in $M \setminus M_2'$, i.e. every point in M left of $\binom{a_2-(b_2-b_1)}{b_1}$ or right of $\binom{a_2-1}{b_2-1}$, is cut off by the constraints $x < 10 \wedge x \geq 5$ (or equivalently $x \leq 9 \wedge x \leq 4$) along the path. This implies $a_2 - (b_2 - b_1) \leq 5$ and $a_2 - 1 \geq 9$.

Example



All these constraints can be combined to $a_1 \leq 0$, $b_1 = 0$, $a_2 \geq 10$, and $a_2 - b_2 = 5$.
Every instantiation of the parameters that satisfies these constraints leads to a valid loop invariant.
To find the strongest invariant, maximize a_1 and b_1 and minimize a_2 and b_2, just as we before maximized the l_i and minimized the u_i. This yields the max-plus polyhedron generated by the two points $\binom{0}{5}$ and $\binom{10}{10}$.

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- An implementation is in progress.